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EUCLIDEANESS AND FINAL POLYNOMIALS IN ORIENTED MATROID THEORY

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This paper deals with a geometric construction of algebraic non-realizability proofs for certain oriented matroids. As main result we obtain an algorithm which generates a (bi-quadratic) final polynomial [3], [5] for any non-euclidean oriented matroid. Here we apply the results of Edmonds, Fukuda and Mandel [6], [7] concerning non-degenerate cycling of linear programs in non-euclidean oriented matroids.

1. Introduction

One of the most important problems in oriented matroid theory is to find good algorithms that decide whether a given oriented matroid is realizable or not. It is the objective of the present paper to provide a link between two different approaches to the above problem: the geometric approach of Edmonds, Mandel and Fukuda [6], [7] and the algebraic approach of Bokowski, Richter and Sturmfels [3], [4], [5].

One algorithmic way to derive a non-realizability proof for an oriented matroid χ is to decide, whether it admits a bi-quadratic final polynomial; if so, χ is not realizable [3]. In fact this way of deriving a non-realizability proof is very effective. As main algorithmic step a LP-problem has to be solved. The number of arithmetic operations needed to solve this special LP-problem lies in $O(n^{6d})$; here d denotes the rank and n the number of elements of χ (compare [8]). On the other hand the realization problem for oriented matroids is known to be NP-hard (compare [9]), so it is very unlikely that the above method works in general. Nevertheless up to now, no non-realizable oriented matroid is known, that does not admit a bi-quadratic final polynomial.

Another way of finding non-realizability proofs for oriented matroids comes from the oriented matroid version of linear programming (compare [2]). Fukuda gave an example of an oriented matroid program that admits a cycling sequence of strictly increasing pivot steps [6]. This cannot happen in the realizable case. So finding such a strictly increasing LP-cycling is another way to prove that a certain oriented matroid is not realizable.

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In this paper we prove that for any oriented matroid program that admits a non-degenerate pivot cycling the underlying oriented matroid admits a bi-quadratic final polynomial.

Since an oriented matroid is non-euclidean if and only if it does not admit a non-degenerate pivot cycle [6] we obtain that any non-realizable oriented matroid that admits no bi-quadratic final polynomial has to be euclidean.

For matters of simplicity we restrict ourselves in this paper to the uniform case only. All results can be generalized to the non-uniform case. A proof that also covers the non-uniform case can be found in [8].

2. Bi-quadratic final polynomials

In this chapter a brief description of the method of bi-quadratic final polynomials will be given. For a detailed discussion of this method the reader is referred to [3]. First we fix our terminology for oriented matroids.

In this section $E_n := \{1,\dots,n\}$ stands for a finite ordered set and $X := (x_1,\dots,x_n) \in R^{nd}$ stands for the representative matrix of a point configuration in R^d with row vectors $x_1,\dots,x_n$. We abbreviate $[\lambda_1,\dots,\lambda_d]_X := det(x_{\lambda_1},\dots,x_{\lambda_d})$. It is well known, that the map $\chi_X : E_n^d \to \{-1,0,+1\}$ defined by $\chi_X(\lambda_1,\dots,\lambda_d) := sign[\lambda_1,\dots,\lambda_d]_X$ corresponds to the basis orientation of the (linear) oriented matroid associated to X.

The set of all formal brackets of rank d on an ordered set E is defined by

$$B_{E,d} := \{[\lambda_1,\dots,\lambda_d] \mid \lambda_1,\dots,\lambda_d \in E\}$$

In the former these brackets will be used as variables simulating the behavior of determinants in the realizable case. We are especially interested in certain polynomials that can be expressed in these variables. For this purpose we define a polynomial ring in which our brackets behave according to the alternating determinant rule.

Let $R_{E,d} := K[B_{E,d}]$ be the integer polynomial ring in all brackets and let $I_{E,d}$ be the following ideal

$$I_{E,d} := \{[\lambda_1,\dots,\lambda_d] - sign(\pi)\cdot[\lambda_{\pi(1)},\dots,\lambda_{\pi(d)}] \mid \lambda \in E^d; \pi \in S_d\}.$$

In the polynomial ring $A_{E,d} := R_{E,d}/I_{E,d}$ the brackets behave according to the alternating determinant rule. We have

$$[\dots,i,\dots,j,\dots] = -[\dots,j,\dots,i,\dots]$$

in $A_{E,d}$. If p is a polynomial in $A_{E,d}$, p_X stands for the evaluation of p for a certain point configuration $X \in R^{nd}$ where the variables $[\lambda_1,\dots,\lambda_d]$ are replaced by the evaluations $[\lambda_1,\dots,\lambda_d]_X$.

For a given $\tau := (\tau_1,\dots,\tau_{d-2}) \in E^{d-2}$ and $\lambda := (\lambda_1,\dots,\lambda_4) \in E^4$ we call the formal expression

$$\begin{aligned}\{\tau|\lambda\} := &+[\tau,\lambda_1,\lambda_2][\tau,\lambda_3,\lambda_4]\\ &-[\tau,\lambda_1,\lambda_3][\tau,\lambda_2,\lambda_4]\\ &+[\tau,\lambda_1,\lambda_4][\tau,\lambda_2,\lambda_3]\end{aligned}$$

a Grassmann–Plücker polynomial in the polynomial ring $A_{E,d}$. As an example, the evaluation of this polynomial for a certain point configuration X will be abbreviated by

$$\begin{aligned}\{\tau|\lambda\}_X := &+[\tau,\lambda_1,\lambda_2]_X[\tau,\lambda_3,\lambda_4]_X\\ &-[\tau,\lambda_1,\lambda_3]_X[\tau,\lambda_2,\lambda_4]_X\\ &+[\tau,\lambda_1,\lambda_4]_X[\tau,\lambda_2,\lambda_3]_X\end{aligned}$$

For a point configuration X and any $\tau \in E^{d-2}, \lambda \in E^4$ we have $\{\tau|\lambda\}_X = 0$ (compare [5]). As a consequence of this equation we can conclude, that the set

$$\begin{aligned}GR(\chi_X,\{\tau|\lambda\}) := \{ &+\chi_X(\tau,\lambda_1,\lambda_2)\chi_X(\tau,\lambda_3,\lambda_4),\\ &-\chi_X(\tau,\lambda_1,\lambda_3)\chi_X(\tau,\lambda_2,\lambda_4),\\ &+\chi_X(\tau,\lambda_1,\lambda_4)\chi_X(\tau,\lambda_2,\lambda_3)\}\end{aligned}$$

either equals $\{0\}$ or contains $\{-1,+1\}$. In fact this is true for all oriented matroids (compare [5]). Moreover this property can be used (together with the underlying matroid structure) as a formal definition for oriented matroids.

Lemma 2.1. *Let χ be a rank d oriented matroid on E. Then for any $\tau \in E^{d-2}, \lambda \in E^4$ we have $GR(\chi_X,\{\tau|\lambda\})$ either equals $\{0\}$ or contains $\{-1,+1\}$.*

We are now ready to define *bi-quadratic inequalities*. It will turn out later that the system of all these inequalities is solvable in the realizable case.

Definition 2.2 Let χ be a rank d uniform oriented matroid on a finite set E of cardinality $n > d$, let $\tau \in E^{d-2}, \lambda \in E^4$ staisfying $|\lambda \cup \tau| = d+2$ and let

$$\begin{aligned}A &:= (\tau,\lambda_1,\lambda_2), & B &:= (\tau,\lambda_3,\lambda_4),\\ C &:= (\tau,\lambda_1,\lambda_3), & D &:= (\tau,\lambda_2,\lambda_4),\\ E &:= (\tau,\lambda_1,\lambda_4), & F &:= (\tau,\lambda_2,\lambda_3).\end{aligned}$$

(1) (τ,λ) is called <u>χ-normalized</u> if $\chi(A)\cdot\chi(B) > 0, \chi(C)\cdot\chi(D) > 0, \chi(E)\cdot\chi(F) > 0$.
(2) For a χ-normalized pair (τ,λ) we call

$$\langle A,B|C,D\rangle \text{ and } \langle E,F|C,D\rangle \quad \underline{\text{bi-quadratic inequalities}}.$$

Note that for any $\tau \in E^{d-2}$ and $\lambda \in E^4$ there is always a suitable permutation π of the elements in λ such that $(\tau,\pi(\lambda))$ is χ-normalized. The set of all bi-quadratic inequalities of χ will be denoted by $\mathscr{B}_\chi$. The bi-quadratic inequalities can be thought of as natural consequences of Grassmann–Plücker Relations in the realizable case.

Remark 2.3. For a point configuration X and its corresponding oriented matroid χ_X we have

$$[A]_X \cdot [B]_X < [C]_X \cdot [D]_X \text{ for all } \langle A,B|C,D\rangle \in \mathscr{B}_{\chi_X}.$$

Now we are ready to give a definition of what we mean by a *bi-quadratic final polynomial*.

Definition 2.4 A uniform oriented matroid admits a bi-quadratic final polynomial if there is a non-empty collection of bi-quadratic inequalities

$$\langle A_i, B_i | C_i, D_i \rangle \in \mathscr{B}_\chi; \quad 1 \leq i \leq k$$

such that the following equality holds in the ring $A_{E,d}$:

$$\prod_{i=1}^{k} [A_i] \cdot [B_i] = \prod_{i=1}^{k} [C_i] \cdot [D_i].$$

This definition allows the following claim (compare Lemma 2.1, [3]).

Lemma 2.5. *If χ admits a biquadratic final polynomial, then χ is not realizable.*

Proof. Assume on the contrary that χ admits a bi-quadratic final polynomial as defined above, and χ is realizable, i.e $\chi = \chi_X$ for a suitable point configuration X. By Remark 2.3. we have

$$[A_i]_X \cdot [B_i]_X < [C_i]_X \cdot [D_i]_X \text{ for all } 1 \leq i \leq k.$$

By definition the products on the left side are all positive and the products on the right side are positive as well. If we multiply all right and all left sides we obtain:

$$\prod_{i=1}^{k} [A_i]_X \cdot [B_i]_X < \prod_{i=1}^{k} [C_i]_X \cdot [D_i]_X.$$

On the other hand the fact that we have a final polynomial implies

$$\prod_{i=1}^{k} [A_i]_X \cdot [B_i]_X = \prod_{i=1}^{k} [C_i]_X \cdot [D_i]_X.$$

This contradicts our assumption. ∎

We have seen that finding a bi-quadratic final polynomial gives us a method to prove non-realizability of uniform oriented matroids. At the first sight it seems that this method is rather special. It seems that when replacing the three-summand Grassmann–Plücker relations $\{\tau|\lambda\} = 0$ by two inequalities a lot of information is lost. Interesting enough up to now no non-linear oriented matroid is known that has definitely no bi-quadratic final polynomial. When searching for such an example one way is to check whether other methods providing non-realizability imply the existence of a bi-quadratic final polynomial. In the following chapter the method of finding non-degenerate cycles in oriented matroid programs first discussed by Fukuda [6] will be outlined. After this we prove that the existence of such a cycling implies the existence of a bi-quadratic final polynomial.

3. Oriented matroid programming

Bland [2] developed oriented matroid theory as a combinatorial abstraction of linear programming. Concepts like *feasible region, objective function, optimality* and *pivot step* have their one to one correspondence in the setting of oriented matroid programming. Purely combinatorial analogues for the simplex algorithm were given. Nevertheless in the more abstract context of oriented matroid programming things can happen which are impossible in the realizable (i.e. linear) case. In [6] Fukuda gave an example of a non-linear oriented matroid with the strange property that when applying the usual simplex algorithm to it, one would run into a *non-degenerate cycling*. In other words there is a sequence of strictly increasing pivot operations that after finitely many steps reaches the starting vertex again. This is impossible in the realizable case. Finding such a non-degenerate cycling for an oriented matroid is a method to prove its non-realizabilty. In this chapter a brief outline of oriented matroid programming will be given. For details the reader is referred to [1], [2], [6].

For the rest of this chapter we assume that χ is an oriented matroid of rank d on the set $E := E_n \cup \{f, g\}$ without loops, coloops and parallel elements. We assume that we have a linear order on E according to $1 < 2 < \cdots < n < f < g$. The triple (χ, f, g) is called an *oriented matroid program.* We will set our notation such that g plays the role of a hyperplane at infinity and such that f will play the role of our objective function.

For a signed vector $X \in \{-1, 0, +1\}^E$ and a subset $F \subseteq E$ the restriction of X on F is denoted by X_F. X_e; $e \in E$ stands for the e-th component of X. The *composition* $X \circ Y$ of signed vectors X and Y is defined by:

$$(X \circ Y)_e := \begin{cases} X_e & \text{if } X_e \neq 0; \\ Y_e & \text{otherwise.} \end{cases}$$

The zero-vector is simply denoted by 0 and the expressions $X = 0$, $X < 0$, $-X$ are interpreted componentwise.

We denote the set of cocircuits of χ by

$$\mathcal{O} := \mathcal{O}(\chi) := \{(\chi(\lambda, 1), \ldots, \chi(\lambda, g)) | \lambda \in E^{d-1}\}.$$

The cocircuit span of χ will be denoted by

$$\mathcal{C} := \mathcal{C}(\chi) := cl(\mathcal{O}(\chi)),$$

where *cl* stand for the closure operator under composition. The elements of the cocircuit span correspond to the cells of the pseudohyperplane arrangement associated with χ. The *affine space* $\mathcal{A}$ with respect to the the infinite pseudohyperplane g is defined by

$$\mathcal{A} := \{X \in \mathcal{C} | X_g > 0\}.$$

The *infinite space* (compare [6]) $\mathcal{A}^\infty$ w.r.t. g is defined by

$$\mathcal{A}^\infty := \{Z \in \mathcal{C} | Z_g = 0\}.$$

Following Fukuda we call an element $X \in \mathcal{A}$ a *solution* and an element $Z \in \mathcal{A}^\infty$ a *direction*. A cocircuit in $\mathcal{A}$ will be called an *affine vertex*. The idea of oriented matroid programming is now to encode a polytope (feasible region) by a fulldimensional cell of $\mathcal{A}$ and to define optimality in terms of the *increase* of directions with respect to the orientation of f. We will skip all these details and come directly to the notion of a non-degenerate cycle on the vertices of $\mathcal{A}$.

Definition 3.1. Let χ be an oriented matroid program on $E := \{1,\ldots,f,g\}$ and $\mathcal{A}$ ($\mathcal{A}^\infty$) the affine (infinite) space with respect to g.

(1) A set $B = \{\lambda_1,\ldots,\lambda_{d-1}\} \in E - \{f,g\}$, such that $B \cup \{g\}$ is independent, is called an <u>affine basis</u>. The unique affine vertex X with $X_B = 0$ will be denoted by $v(B)$.

(2) $B_1 \to B_2$ is called a <u>pivot operation</u> if B_1, B_2 are affine bases and $L := B_2 - \{b\} = B_1 - \{a\}$ for certain elements $a, b \in E - \{f,g\}; a \neq b$. L is called the <u>edge</u> of $B_1 \to B_2$

(3) The <u>direction</u> of a pivot $L \cup \{a\} = B_1 \to B_2 = L \cup \{b\}$, where $L \cup \{a,b\}$ is assumed to be independent, is the unique vector $d := d(B_1 \to B_2) \in \mathcal{A}^\infty$ with $d_L = 0$ and $d_a = v(B_2)_a$.

(4) A pivot operation $B_1 := L \cup \{a\} \to L \cup \{b\} =: B_2; a \neq b$ is called
<u>degenerate</u> if $v(B_1) = v(B_2)$,
<u>horizontal</u> if $L \cup \{f,g\}$ is dependent,
<u>strictly increasing</u> if $d(B_1 \to B_2)_f > 0$ and $B_1 \to B_2$ is not degenerate.

In Figure 1. the situation is demonstrated for a rank 3 oriented matroid program. Here the corresponding pseudoline arrangement for an oriented matroid program on the elements $1,2,3,4,f,g$ is given. In the example $\{1,2\} \to \{1,3\}$ is a degenerate pivot operation, $\{1,2\} \to \{1,4\}$ is strictly increasing and $\{1,4\} \to \{3,4\}$ is horizontal. All strictly increasing pivot operations are marked by arrows and all affine vertices are marked by dots.

Now we define a non-degenerate cycle on an oriented matroid program χ over $E_n \cup \{f,g\}$.

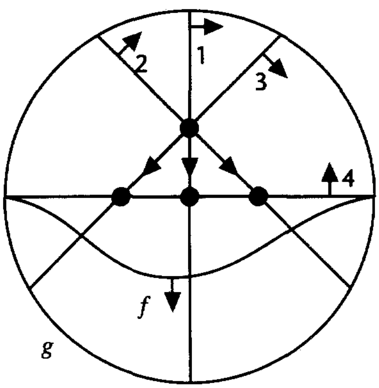

Fig. 1. Example of an oriented matroid program

Definition 3.2. A sequence of pivot operations:

$$B_1 \to B_2 \to \cdots \to B_k$$

is called a non-degenerate cycle on χ if $B_1 = B_k$ and all pivot operations are either degenerate, horizontal or strictly increasing and at least one pivot is strictly increasing.

In [6] Fukuda proved the following result:

Lemma 3.3. *An oriented matroid program admits a non-degenerate cycle if and only if it is non-euclidean.*

As a direct consequence we obtain:

Corollary 3.4. *Whenever an oriented matroid program admits a non-degenerate cycle, the underlying oriented matroid is not realizable.*

Fukuda gave furthermore a minimal example on 8 elements in rank 4 where such a situation occurs. In the meantime all uniform oriented matroids on 8 elements in rank 4 had been classified by Bokowski and myself [4]. It turned out, that exactly 24 of the corresponding reorientation classes (i.e. unlabeled pseudohyperplane arrangements) are non-realizable, 19 of which admit a non-degenerate cycle. All 24 reorientation classes admit a bi-quadratic final polynomial. In the next section we will give a constructive method, that generates a bi-quadratic final polynomial whenever a non-degenerate cycle is given.

4. How to obtain bi-quadratic final polynomials from non-degenerate LP-cycles

We now restrict ourselves again to the uniform case. For uniform oriented matroid programs no horizontal or degenerate pivot steps can occur. The proof of our main result will be divided into two steps. First we will translate the concept of *strictly increasing* pivot steps in terms of sign conditions for the bases of our oriented matroid program. Afterwards these sign conditions will be used to study the dependencies between Grassmann–Plücker relations and non-degenerate cycles. As a first result we obtain

Lemma 4.1. *Let (χ, f, g) be an uniform oriented matroid program as defined above. Let $L := \{\lambda_1, \dots, \lambda_{d-2}\} \subset E - \{f, g\}$, and $a, b \in E - \{f, g\}$ such that*

$$L \cup \{a\} = B_1 \to B_2 = L \cup \{b\}$$

is a pivot operation with edge L. If $B_1 \to B_2$ is strictly increasing then

$$\chi(\lambda_1, \dots, \lambda_{d-2}, g, f) \cdot \chi(\lambda_1, \dots, \lambda_{d-2}, a, b) \cdot$$
$$\cdot \chi(\lambda_1, \dots, \lambda_{d-2}, g, a) \cdot \chi(\lambda_1, \dots, \lambda_{d-2}, g, b) = +1.$$

Proof. Let $B_1 \to B_2$ be strictly increasing. Since the pivot operation is not degenerated, the set $L \cup \{a, b\}$ is independent. Let $d := d(B_1 \to B_2)$ be the (well defined) direction of the pivot operation. The following relations hold:

$$(1) \qquad v(B_1)_g = +1; v(B_1)_b = \sigma_1; v(B_1)_a = 0,$$
$$(2) \qquad v(B_2)_g = +1; v(B_2)_a = \sigma_2; v(B_2)_b = 0,$$
$$(3) \qquad d_a = \sigma_2; d_f := +1; d_g = 0.$$

Here σ_1 and σ_2 are signs depending on the underlying oriented matroid. The first two statements follow from the definition of the $v(B_i)$, the third statement is a consequence of the fact that d is the direction of $B_1 \to B_2$.

If we apply these statements to the bases orientations of χ we obtain the following sequence of equalities (we abbreviate $\lambda := (\lambda_1, \ldots, \lambda_{d-1})$):

$$\chi(\lambda, g, f) \overset{(3)}{=} \sigma_2 \cdot \chi(\lambda, g, a) \overset{(1)}{=} \sigma_2 \cdot \sigma_1 \cdot \chi(\lambda, b, a) \overset{(2)}{=} \sigma_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \chi(\lambda, b, g)$$

Using this equation we obtain:

$$\begin{aligned}&\chi(\lambda, g, f) \cdot \chi(\lambda, a, b) \cdot \chi(\lambda, g, a) \cdot \chi(\lambda, g, b) = \\ &\qquad \chi(\lambda, g, f) \cdot \sigma_2 \cdot \sigma_1 \cdot (-\chi(\lambda, g, f)) \cdot \sigma_2 \cdot \chi(\lambda, g, f) \cdot \sigma_1 \cdot (-\chi(\lambda, g, f)) = 1.\end{aligned}$$

This proves the claim. ∎

We are now ready to prove the main theorem:

Theorem 4.2. *Let (χ, f, g) be an uniform oriented matroid program as defined above. If (χ, f, g) admits a non-degenerate cycle then χ also admits a bi-quadratic final polynomial.*

Proof. Let χ be a rank d uniform oriented matroid program on the set $E := \{1, \ldots, n, f, g\}$. For a given non-degenerate cycle

$$B_1 \to B_2 \to \ldots \to B_k; \quad B_1 = B_k$$

on χ we will construct a suitable bi-quadratic final polynomial. Formally we add a base $B_{k+1} := B_2$. We define L^i, a^i, b^i according to

$$L^i \cup \{a^i\} := B_i \to B_{i+1} =: L^i \cup \{b^i\} \text{ for all } 1 \le i \le k.$$

$L^i =: \{\lambda^i_1, \ldots, \lambda^i_{d-2}\}$ is the edge of the pivot operation $B_i \to B_{i+1}$. Furthermore we abbreviate $\lambda^i := (\lambda^i_1 \ldots, \lambda^i_{d-2})$. We now consider the following sequence of Grassmann–Plücker polynomials:

$$GP^i := \{\lambda^i | g, f, a^i, b^i\} \text{ for all } 1 \le i \le k.$$

According to our definition we have $GP^1 = GP^k$. We set:

$$\begin{aligned} A^i &:= (\lambda^i, g, f), & B^i &:= (\lambda^i, a^i, b^i), \\ C^i &:= (\lambda^i, g, a^i), & D^i &:= (\lambda^i, f, b^i), \\ E^i &:= (\lambda^i, g, b^i), & F^i &:= (\lambda^i, f, a^i), \end{aligned}$$

such that we have $\{\lambda^i | g, f, a^i, b^i\} = A^i \cdot B^i - C^i \cdot D^i + E^i \cdot F^i$. Notice that $(\lambda^i, (g, f, a^i, b^i))$ is not yet normalized.

Since we had a non-degenerate cycle of an uniform oriented matroid program we can only have strictly increasing pivot operations $B_i \to B_{i+1}$. In this case according to Lemma 4.1. we have $\chi(A^i) \cdot \chi(B^i) \cdot \chi(C^i) \cdot \chi(E^i) = +1$. This relation restricts the possible signs appearing in the Grassmann–Plücker relations. Together with the usual restriction for oriented matroids (Lemma 2.1) only the twelve types of sign structures given in the table below are possible.

$$\underbrace{A^i \cdot B^i} - \underbrace{C^i} \cdot \underbrace{D^i} + \underbrace{E^i} \cdot \underbrace{F^i}$$

$+$	$+$	$+$	$+$	$+$	type 1
$+$	$+$	$+$	$+$	$-$	type 2
$+$	$+$	$-$	$+$	$-$	type 3
$+$	$-$	$-$	$-$	$-$	type 4
$+$	$-$	$-$	$-$	$+$	type 5
$+$	$-$	$+$	$-$	$+$	type 6
$-$	$+$	$-$	$-$	$+$	type 7
$-$	$+$	$-$	$-$	$-$	type 8
$-$	$+$	$+$	$-$	$-$	type 9
$-$	$-$	$+$	$+$	$-$	type 10
$-$	$-$	$+$	$+$	$+$	type 11
$-$	$-$	$-$	$+$	$+$	type 12

Table

After normalization, Grassmann–Plücker relations of type 1, 4, 7 or 10 generate the bi-quadratic inequality $\langle E^i, F^i | C^i, D^i \rangle$ and the Grassmann–Plücker relations of type 4, 6, 9 or 12 generate the bi-quadratic inequality $\langle C^i, D^i | E^i, F^i \rangle$. Now we study the structure of the whole system of Grassmann–Plücker relations GP^i induced by our non-degenerate pivot cycle. By definition we have $L^i \cup \{b^i\} = B_{i+1} = L^{i+1} \cup \{a^{i+1}\}$. This yields

$$\begin{aligned}&\chi(D^i) \cdot \chi(E^i) \cdot \chi(C^{i+1}) \cdot \chi(F^{i+1}) = \\ &\qquad \chi(\lambda^i, f, b^i) \cdot \chi(\lambda^i, g, b^i) \cdot \chi(\lambda^{i+1}, g, a^{i+1}) \cdot \chi(\lambda^{i+1}, f, a^{i+1}) = 1.\end{aligned}$$

This relation restricts the types of possible successors GP^{i+1} of a Grassmann–Plücker relation GP^i of a certain type. The graph of possible successors is symbolized in Figure 2.

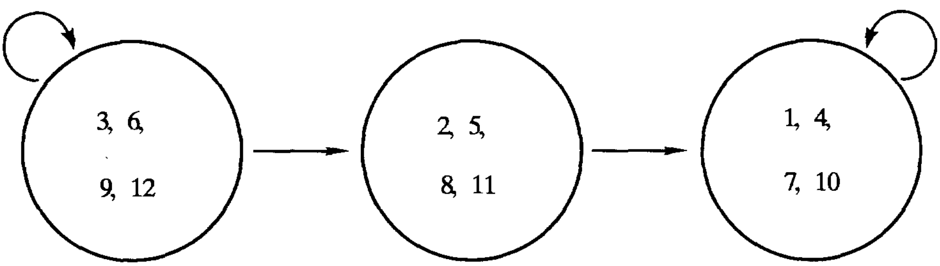

Fig. 2. Graph of possible successors

A Grassmann–Plücker relation of type t can be succeeded by a Grassmann–Plücker relation of type s if and only if there is an arrow from the circle containing t to the circle containing s. We had $GP^1 = GP^k$, so our non-degenerate pivot cycle translates to a circuit in the graph of possible successors of our Grassmann–Plücker relations. Such a circuit can only appear if either all Grassmann–Plücker relations are of types 1, 4, 7 or 10 or all Grassmann–Plücker relations are of types 3, 6, 9 or 12. In both cases the resulting set of biquadratic inequalities gives us a biquadratic final polynomial. This proves the claim. ∎

Finally as a direct consequence of Lemma 3.3. and Theorem 4.2 we obtain:

Corollary 4.3. *Every non-euclidean uniform oriented matroid admits a bi-quadratic final polynomial.*

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